Foliated almost 3-contact manifolds

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The theory of almost 3-contact manifolds was initiated by the Japanese school in the 70 (cf. [7], [9], [10]). An *almost 3-contact structure* on a smooth manifold M is given by three distinct almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) on M, which are related by the following relations

$$\phi_k = \phi_i\phi_j - \eta_j \otimes \xi_i = -\phi_j\phi_i + \eta_i \otimes \xi_j,$$

$$\xi_k = \phi_i\xi_j = -\phi_j\xi_i, \ \eta_k = \eta_i \circ \phi_j = -\eta_j \circ \phi_i,$$

for a cyclic permutation (i, j, k) of $\{1, 2, 3\}$. One can prove then that the dimension of M must be of the form $4n + 3$. Moreover there exists a Riemannian metric g compatible with each of the given almost contact structures, hence one can speak of *almost* 3*-contact metric manifolds*.

Almost 3-contact metric manifolds, and especially some their remarkable subclasses such as 3-Sasakian manifolds, have became subject of interest in recent years due to some physical applications (cf. [6], [11], etc.) and, from the mathematical point of view, to the work on this subject by Boyer, Galicki and their collaborators (cf. [1] and reference therein).

An old question on almost 3-contact geometry is whether the 3-dimensional distribution $\mathcal{V}$ spanned by the Reeb vector fields ξ_1, ξ_2, ξ_3 is integrable (cf. [8]). This holds, for instance, for 3-Sasakian manifolds and 3-cosymplectic manifolds, which are examples of *hyper-normal almost* 3*-contact manifolds*, that is each structure (ϕ_i, ξ_i, η_i) is normal in the sense that $[\phi_i, \phi_i] + d\eta_i \otimes \xi_i = 0$. Therefore one can ask whether there is some relations between the hyper-normality of the almost 3-contact structure and integrability of $\mathcal{V}$.

In this talk we illustrate the negative solution [5] of this open problem, showing an explicit example of hyper-normal almost 3-contact manifold such that $\mathcal{V}$ is not integrable and of non-hyper-normal almost 3-contact manifolds with $\mathcal{V}$ integrable. Moreover, we characterize almost 3-contact manifolds such that $\mathcal{V}$ is integrable, which we call *foliated almost* 3*-contact manifolds*, proving that the integrability of $\mathcal{V}$ is closely related to the property of the Reeb vector fields to be Killing.

Then we pass to study some remarkable subclasses of foliated almost 3-contact manifolds, and in particular 3*-quasi-Sasakian manifolds* ([3], [4]) and *almost* 3*-contact metric manifolds with torsion* ([5]). Many properties and results about these geometric structures are presented, especially those related to the foliation $\mathcal{V}$, and their relation with the their quaternionic analogues is pointed out.

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